

# Non-existence of the final first integral in the Zipoy-Voorhees space-time

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We show that the geodesic motion in the Zipoy-Voorhees space-time is not Liouville integrable in that there does not exist an additional first integral, meromorphic in the phase space variables.

## I. INTRODUCTION

The question of integrability of the test particle motion in the Zipoy-Voorhees metric has recently attracted some attention, with both numerical [1, 2] and analytical investigations [3]. The significance of this problem is the same as in the classical result by Carter [4] that there exists an additional first integral in the Kerr space-time, making the problem completely integrable. Carter's integral is not generated by a Killing vector, so it is not a usual symmetry of the manifold, but it is quadratic in momenta which has important consequences. Such integrals translate into the separability of the Hamilton-Jacobi equation and d'Alembertian [5], which in turn appears in the Teukolsky [6] equation. That is to say, both the classical problem of particle motion in this space-time, the linear perturbation equations governing the gravitational waves and potentially quantum equations in that background become considerably easier to solve. This fact is also used in numerical approaches, when trying to determine possible spectra of gravitational radiation in anticipation of the observed data [7].

It is then natural to analyse other space-times which could serve as models of compact objects, and the stationary axisymmetric ones are one direction to explore. However, despite some numerical evidence [1] we find that the particular Zipoy-Voorhees metric with the parameter  $\delta = 2$  is not integrable. To be more precise, we consider the motion of a test particle as a Hamiltonian system, and ask for existence of an additional constant of motion that would yield *Liouvillian integrability* with respect to the canonical Poisson bracket  $\{\}$ . That is, for all first integrals  $I_k$  we would have  $\{I_k, I_l\} = 0$ , where the Hamiltonian is included as  $H = I_1$ . It turns out, that no such first integral can be found in the class of meromorphic functions, and we have to use the differential Galois theory to prove that.

## II. FORMULATION OF THE PROBLEM

The Zipoy-Voorhees metric under consideration is given by

$$ds^2 = - \left( \frac{x-1}{x+1} \right)^2 dt^2 + \frac{(x+1)^3(1-y^2)}{x-1} d\phi^2 + \frac{(x^2-1)^2(x+1)^4}{(x^2-y^2)^3} \left( \frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2} \right), \quad (1)$$

where  $x$ ,  $y$  and  $\phi$  form the prolate spheroidal coordinates. Instead of working directly with the geodesic equations we take the Hamiltonian approach with

$$H = \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta = - \frac{(x+1)^2}{2(x-1)^2} p_0^2 + \frac{(x^2-y^2)^3}{2(x-1)(x+1)^5} p_1^2 + \frac{(x^2-y^2)^3(1-y^2)}{2(x-1)^2(x+1)^6} p_2^2 + \frac{x-1}{2(x+1)^3(1-y^2)} p_3^2, \quad (2)$$

where the canonical coordinates are

$$q_0 = t, \quad q_1 = x, \quad q_2 = y, \quad q_3 = \phi. \quad (3)$$

The equations then read

$$\begin{cases} \frac{dq_i}{d\tau} = \frac{\partial H}{\partial p_i}, \\ \frac{dp_i}{d\tau} = - \frac{\partial H}{\partial q_i}, \end{cases} \quad (4)$$

and the normalization of four-velocities gives the value of the (conserved) Hamiltonian to be  $H = -\frac{1}{2}\mu^2$ . The new time parameter is the rescaled proper time  $\mu\tau = s$ , which allows us to include the zero geodesics for photons without introducing another affine parameter but with simply  $\mu = 0$ .

Since the metric has two Killing vector fields  $\partial_t$  and  $\partial_\phi$ , the two momenta  $p_0$  and  $p_3$  are conserved. Together with the Hamiltonian they provide three first integrals. The question then is, whether there exists one more first integral that would make the system Liouville integrable. The precise notion is expressed in the following theorem.

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**Theorem 1** *If the Hamiltonian system (4) is Liouville integrable then the identity component of the differential Galois group of the normal variational equations along any non-constant particular solution of (4) is Abelian.*

Let us try to explain the involved mathematics somewhat, although for a proper, detailed exposition of the subject the reader is referred to [8–10], and a short introduction with application to another relativistic system can be found in [11].

The key steps are that we can find a particular solution  $\varphi(\tau)$  of our equations, and then construct the variational equations around it. Writing (4) jointly as  $\dot{u}_i = f_i(u)$ , we define the variational equations as

$$\dot{\xi} = \left. \frac{\partial f_i}{\partial u_j} \right|_{\varphi(\tau)} \xi_j. \quad (5)$$

The normal part is the subspace formed by excluding the variations tangent to the particular solution and projected onto the constant energy hypersurface, which reduces the number of equations by 2. In general, one often changes the independent variable  $\tau \rightarrow z$  for practical reasons so that the normal variational equations (NVE) can be written as

$$\xi'_i(z) = A_i^j(z) \xi_j(z). \quad (6)$$

Since the first integrals are to be meromorphic as functions of positions and momenta, note that we are now dealing with complex functions, and extend also the time to the complex domain. So even though the particular solutions and equations might lie outside the physical (real) domain, the non-existence of complex first integrals will preclude the existence of real ones as well, since the latter are just special cases of the former.

The integrability of the original system translates into solvability of the variational equations, which are much easier to analyse being linear. It is at this level, that we define the group of transformations acting on some function field to which the solutions of the NVE belong. The subgroup which leaves the of the elements of  $A_i^j(z)$  fixed is the differential Galois group  $\mathcal{G}$ .

**Theorem 2** *If the differential Galois group of a linear system (6) of equations has a solvable element of identity, then all the solutions of the system are Liouvillian.*

In other words, if the group in question has an Abelian, hence solvable, identity component, the NVE are solvable with Liouvillian functions. Fortunately, one does not have to check the group directly, and in the case of second order equations with rational coefficients there is an effective algorithm for finding the Liouvillian solutions [12].

The plan of attack is thus to look for particular solutions, for which the NVE has a block structure so that a two dimensional subsystem can be separated. We then rewrite is a second order linear differential equation with

rational coefficients and apply the Kovacic algorithm to see if it has any Liouvillian solutions. Note that the system has no external parameters, and only the values of particular first integrals enter as internal parameters. They are synonymous with initial conditions, so that if we manage to find just one solution, for particular values of  $\mu$ ,  $p_0$  and  $p_3$ , such that the respective NVE is unsolvable, we will have proven that there cannot exist another first integral over the whole phase space.

It might so happen, unlike in the Carter case, that the system exhibits some particular invariant set on which there exists an additional integral. For example one could have  $\dot{I}_4 = H$ , which would mean that  $I_4$  is conserved on the zero energy hypersurface  $\mu = 0$  which is clearly a physically distinguished case. We will then have to look for particular solutions on those sets to make the results even more restrictive than just the lack of a global first integral.

### III. PROOF OF NON-INTEGRABILITY

The obvious particular solution to look at is a particle moving along a straight line, through the center in the equatorial plane, which in prolate coordinates means  $y = 0$  and  $p_3 = 0$ . The non-trivial equations then read:

$$\begin{aligned} \frac{dt}{d\tau} &= -\frac{(x+1)^2 p_0}{(x-1)^2}, \\ \frac{dx}{d\tau} &= \frac{p_1 x^6}{(x-1)(x+1)^5}, \\ \frac{dp_1}{d\tau} &= p_1^2 \frac{x^5(3-2x)}{(x+1)^6(x-1)^2} - p_0^2 \frac{2(x+1)}{(x-1)^3}. \end{aligned} \quad (7)$$

Or, upon rescaling time by

$$d\tau = \frac{(x-1)^2(x+1)^3}{x^3} du, \quad (8)$$

we have

$$\begin{aligned} \dot{x} &= \frac{p_1 x^3(x-1)}{(x+1)^2}, \\ \dot{p}_1 &= p_1^2 \frac{x^2(3-2x)}{(x+1)^3} - p_0^2 \frac{2(x+1)^4}{x^3(x-1)}, \end{aligned} \quad (9)$$

where the dot denotes differentiation with respect to  $u$ , and we have omitted the  $t$  equation, as the other two separate from it. This is the two-dimensional subsystem of the NVE that was mentioned before. The conservation of the Hamiltonian now reads

$$-\frac{1}{2}\mu^2 = -\frac{p_0^2(x+1)^8 + p_1^2 x^6(1-x^2)}{2(x-1)^2(x+1)^6}, \quad (10)$$

which together with the equation for  $\dot{x}$  yields

$$\dot{x}^2 = (x^2-1)(p_0^2(x+1)^2 - \mu^2(x-1)^2), \quad (11)$$

so that  $x(u)$  is expressible by the Jacobi elliptic functions. This fact is important, as we will change the independent variable from  $u$  to  $x$  which is permissible (does not change the identity component of the Galois group) only if the function  $x(u)$  defines a finite cover of the complex plane [9].

The variational equations along this solution separate so that the NVE read

$$\begin{aligned}\dot{\xi}_1 &= \frac{x^3}{(1+x)^3} \xi_2, \\ \dot{\xi}_2 &= 3p_1^2 \frac{x(x-1)}{(x+1)^2} \xi_1,\end{aligned}\tag{12}$$

and can be brought to the standard form of

$$\begin{aligned}\xi_2''(x) &= \frac{R(x)}{4x^2(x^2-1)^2(p_0^2(x+1)^2 - \mu^2(x-1)^2)^2} \xi_2(x) \\ &=: r(x) \xi_2(x),\end{aligned}\tag{13}$$

where  $R$  is the following polynomial

$$\begin{aligned}R(x) &= p_0^4(34x^2 - 40x + 3)(x+1)^4 \\ &\quad - 6p_0^2\mu^2(6x^2 - 10x + 1)(x^2-1)^2 \\ &\quad + \mu^4(22x^2 - 20x + 3)(x-1)^4.\end{aligned}\tag{14}$$

Since for all physical particles we have  $p_0 \neq 0$  all the others parameters can be rescaled by it ( $\mu \rightarrow \mu/p_0$ ,  $p_3 \rightarrow p_3/p_0$ ), which we use in what follows.

### A. General $r(x)$

The poles of  $r(x)$  are

$$\left\{-1, 0, 1, \frac{\mu-1}{\mu+1}, \frac{\mu+1}{\mu-1}\right\},\tag{15}$$

and for all of them to be different we must have  $\mu^2 \neq 1$  and  $\mu \neq 0$ . All are of order 2, and the order at infinity is 4, so that we need to check all the cases of the Kovacic algorithm.

In case 1, the appropriate quantities  $\alpha_c^\pm$  form the following set

$$\left\{(0, 1), \left(\frac{9}{4}, -\frac{5}{4}\right), \left(\frac{3}{2}, -\frac{1}{2}\right), \left(\frac{3}{4}, \frac{1}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)\right\},\tag{16}$$

(the first pair corresponding to  $\infty$ ), and the combinations

$$d = \alpha_\infty^\pm - \sum_{c,s} \alpha_c^s\tag{17}$$

give only nine non-negative integers (not all distinct) as possible degrees of the appropriate polynomial  $P$ . However the respective test solutions require that  $\mu = 0$ , and have to be discarded so that this case cannot hold.

In case 2, the families  $E_c$  are

$$\begin{aligned}\{(0, 2, 4), (9, 2, -5), (6, 2, -2), \\ (3, 2, 1), (5, 2, -1), (5, 2, -1)\},\end{aligned}\tag{18}$$

which in turn give 131 possible integer degrees for the polynomial  $P$ . Checking them one by one, we find that they require  $\mu = \pm 1$  in order to form a solution, so that this case can be discarded as well under the current assumptions.

In case 3, the families  $E_c$  contain  $6 \times 13 = 78$  numbers, which make 4826809 combinations for  $d$  out of which 230856 are non-negative integers. We thus first, resort to checking for the presence of logarithms in the solutions, which would prevent this case [9].

The only poles with integer difference in the exponents are 0 and  $\infty$ . Using the Frobenius method, we get the two independent solutions around zero

$$\begin{aligned}v_1 &= x^{3/2} \left(1 + \frac{5(\mu^2-2)}{3(1-\mu^2)}x + \frac{23\mu^4-38\mu^2+65}{12(1-\mu^2)^2}x^2 + \dots\right), \\ v_2 &= \ln(x)v_1 + x^{-1/2} \left(\frac{\mu^2-1}{9(5-\mu^2)} + \frac{5(\mu^2-2)}{9(5-\mu^2)}x + \dots\right).\end{aligned}\tag{19}$$

As can be seen, they require that  $\mu^2 \neq 5$ , and when the equality holds the coefficients in the second series have to be recalculated to give

$$\begin{aligned}v_1 &= x^{3/2} \left(1 - \frac{5}{4}x + \frac{75}{32}x^2 + \dots\right) \\ v_2 &= x^{-1/2} \left(1 + \frac{15}{4}x + \frac{1485}{64}x^2 + \dots\right).\end{aligned}\tag{20}$$

The solutions around  $\infty$  do not have logarithms at all, which means that the only possibility for case 3 left here is with  $\mu^2 = 5$ .

### B. Special subcases

In order to exclude the special energy hypersurfaces we have to resort to a more general particular solution, namely one with  $p_3 \neq 0$ . As already mentioned, it is enough to find one solution for each such surface, and that means we can take a specific value of  $p_3$ . The corresponding NVE will only have numeric coefficients, and checking for its Liouvillian solutions is much easier, for it suffices to use the “kovacicsols” routine implemented in Maple [].

The solution will also be expressible by (hyper-)elliptic function as defined by the Hamiltonian constraint

$$\dot{x}^2 = \frac{(x-1)(x+1)^5 - (x-1)^4 p_3^2 - (x^2-1)^3 - \mu^2}{(x+1)^2},\tag{21}$$

and the counterpart of the NVE (12) will read

$$\begin{aligned}\dot{\xi}_1 &= \frac{x^3}{(1+x)^3} \xi_2, \\ \dot{\xi}_2 &= \frac{(x-1)(3p_1^2 x^4 - (x^2-1)^2 p_3^2)}{x^3(x+1)^2} \xi_1,\end{aligned}\quad (22)$$

We then proceed exactly as above, taking  $x$  as the independent variable and reducing the system to one equation of the form  $\xi_2'' = r\xi$ . For each hypersurface in question, the value of  $p_3 = 1$  leads to a NVE that is not solvable with Liouvillian functions. This finishes the proof for all possible levels of the Hamiltonian.

To further illustrate the complexity of this system, we have also obtained a Poincaré section for the plane  $y = 0$  shown in Fig. 1. The numerical integrator was based on the Bulirsch-Stoer modified midpoint scheme with Richardson extrapolation. We note that the special solution (11) lies entirely in the plane  $y = 0$  and is a trajectory beginning and ending at the singularity so it does not contribute to the section. It also lies outside the visible chaotic region, which is confined to a very small subset of the phase space, as mentioned in [2].

#### IV. GENERAL METRIC

The above results carry, to some extent, to the general Zipoy-Voorhees metric given by

$$\begin{aligned}ds^2 &= -\left(\frac{x-1}{x+1}\right)^\delta dt^2 + \left(\frac{x+1}{x-1}\right)^\delta \left( (x^2-1)(1-y^2)d\phi^2 \right. \\ &\quad \left. + \left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2} (x^2-y^2) \left( \frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2} \right) \right).\end{aligned}\quad (23)$$

The main problem that arises for arbitrary  $\delta$  is that the special solution might no longer be a (hyper-)elliptic function, because the Hamiltonian now gives

$$\begin{aligned}\dot{x}^2 &= \frac{1}{x^2} \left(1 - \frac{1}{x^2}\right)^{-\delta^2} (x+1)^{-2\delta} \left( (x+1)^{2\delta} (x^2-1) p_0^2 \right. \\ &\quad \left. - (x-1)^{2\delta} p_3^2 - (x^2-1)^{\delta+1} \mu^2 \right),\end{aligned}\quad (24)$$

so the right hand side is not necessarily a polynomial or rational function. Accordingly, the rationalization of the NVE might not preserve the identity component of the differential Galois group. However, when  $\delta$  is rational we can still proceed by taking a new dependent variable to be

$$w := \frac{x+1}{x-1}, \quad (25)$$

as this leads to the normal form (13) which involves only integral powers of  $w$  and  $w^\delta$ . Assuming then that  $\delta = p/q$ , we can make the NVE rational by taking  $w^{1/q}$  as the new variable if need be. Unfortunately, the number of poles (and their values) now depends on  $p$  and  $q$ , so the Kovacic algorithm has to be applied to each  $\delta$  separately, but in each of them it is as straightforward as above to use the Maple package, once suitable numeric values of the parameters have been chosen. For example, we have verified that  $\delta = 1/2$  is also non-integrable, confirming the numerical evidence of [2] that both  $\delta > 1$  and  $\delta < 1$  do not admit additional first integrals.

#### V. CONCLUSIONS

Our main result can be stated as

**Theorem 3** *There does not exist an additional, meromorphic first integral of the geodesic motion in the Zipoy-Voorhees metric (1).*

This confirms the previous considerations of [3], and goes much further than first integrals polynomial in momenta. Meromorphic functions include not only the analytic functions of both momenta and coordinates, but also rational and transcendental ones as long as their singularities are just poles. This result thus strongly reduces the possibility of using constants of motion expansion in solving the equations of geodesic motion or gravitational waves.

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Figure 1. Poincare section for the system (2) at  $y = 0$ . The parameters values were:  $p_0 = 0.95$ ,  $p_3 = 3$ ,  $\mu = 1$ .

